Comment on the heavy → light form factors

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Abstract:

We propose a simple parametrization for the form factors in heavy to light decays which satisfies the heavy quark scaling laws and avoids introducing the explicit "dipole" form. The illustration is provided on the set of lattice data for $B \to \pi$ semileptonic decay. The resulting shape of the form factor also agrees with the QCD sum rule predictions.

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1. <u>Preliminaries</u>   A lack of the precise information about the shapes of various form factors from the first principles of QCD was, and still is the main source of uncertainties in extraction of the CKM parameters from the experimentally measured exclusive decay modes [1]. This problem is particularly pronounced in the *heavy → light* processes because the kinematically accessible region is very large. In such a situation it is not only important to know the absolute value of a particular form factor at one specific point (traditionally at $q^2 = 0$), but also its behaviour in q^2, which (when suitably integrated over the whole phase space) gives the physically measurable branching ratio. So far, there were very many studies devoted to that issue. In spite of the recent progress, there is still no method based on QCD only, which could be used to describe the complicated nonperturbative dynamics in the whole physical region[1]. Most of the approaches agree that the functional dependence of the *heavy → light* form factors is 'distorted' in the region of not so high q^2 (or equivalently, for moderately and very large transfers, $\vec{q}^2$). In other words, it is different from what one would obtain by invoking the nearest pole dominance. Although somewhat intriguing, this discrepancy with the pole dominance hypothesis is however expected: the low q^2 region is very far from the first pole, so that many singularities (higher excitations and multiparticle states) become 'visible' too, *i.e.* their contribution becomes sensible (comparable to that of the first pole). One would obviously like to quantify these effects of higher states.

For definiteness, we consider the form factors $f^{+,0}(q^2)$ which parametrize $B \to \pi \ell \nu$ decay as

$$\langle \pi(p')|V_\mu(0)|B(p)\rangle = \left(p + p' - q\frac{m_B^2 - m_\pi^2}{q^2}\right)_\mu f^+(q^2) + \frac{m_B^2 - m_\pi^2}{q^2}q_\mu \, f^0(q^2) \,, \quad (1)$$

where $V_\mu = \bar{u}\gamma_\mu b$. $f^+(q^2)$ is the dominant form factor, *i.e.* the only one needed for the decay rate

$$\frac{d\Gamma}{dq^2}(\bar{B}^0 \to \pi^+\ell^-\bar{\nu}_\ell) = \frac{G_F^2|V_{ub}|^2}{192\pi^3 m_B^3} \, \lambda^{3/2}(q^2) \, |f^+(q^2)|^2 \,, \quad (2)$$

in the case of a massless lepton[2] ($\ell = e, \mu$). The subdominant form factor $f^0(q^2)$, measures the divergence of the vector current and its contribution is proportional to m_ℓ^2 in the decay rate, thus it is irrelevant for present experiments [1]. Its knowledge is however important in lattice analyses, as well as in the approaches relying on the soft pion theorem (in which the form factors are calculated at large q^2), because it can help to constrain the dominant $f^+(q^2)$, through the kinematic condition

$$f^+(0) = f^0(0) \quad . \quad (3)$$

2. <u>Constraints</u>   Beside this important constraint (3), in the heavy quark limit ($m_B \to \infty$) and near the zero-recoil point ($\vec{q}^2 \simeq 0$, or equivalently $q^2 \simeq q_{\max}^2$) where the pion is soft,

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[1]The recent reviews about the form factors in heavy to light decays can be found in Refs. [2, 3]. They also contain an exhaustive list of references.

[2] $\lambda(t) = (t + m_B^2 - m_\pi^2)^2 - 4m_B^2 m_\pi^2$, is the usual triangle function.

one encounters the well known Isgur-Wise scaling law [4][3]

$$\langle \pi(p')|V_\mu|B(p)\rangle \propto \sqrt{m_B} \left(1 + \mathcal{O}(1/m_B) \right) ,$$

which in terms of form factors reads:

$$f^+(q^2 \simeq q^2_{\max}, m_B) \sim \sqrt{m_B} , \quad f^0(q^2 \simeq q^2_{\max}, m_B) \sim \frac{1}{\sqrt{m_B}} . \tag{4}$$

As first noticed in Ref. [5], one may use the low energy theorem to get one more constraint, often invoked in $K_{\ell3}$ decays. Namely, in the soft pion limit ($p'_\mu \to 0$ and $m^2_\pi \to 0$), the r.h.s. of (1) simply becomes $f^0(p^2)p_\mu$, while the l.h.s. from the LSZ reduction of the pion amounts to $(f_B/f_\pi)\,p_\mu$, so that one deduces:

$$f^0(m^2_B) = \frac{f_B}{f_\pi} . \tag{5}$$

Next scaling law comes from the asymptotic limit of the light cone QCD sum rules which are applicable in the low q^2 region. It was first shown in Ref. [6], that the form factors in heavy to light decays satisfy

$$f^{+0}(q^2 \simeq 0, m_B) \sim m_B^{-3/2} . \tag{6}$$

This scaling law was recently derived also in the framework of the large energy effective theory [7]. (6) will provide us an important additional constraint in what follows.

Our knowledge of the form factors goes a little bit beyond. We know that the Lorentz invariant form factors are analytic functions satisfying the dispersion relations:

$$f^0(q^2) = \frac{1}{\pi} \int_{t_0}^\infty dt \frac{\mathrm{Im}\, f^0(t)}{t - q^2 - i\epsilon} , \tag{7}$$

$$f^+(q^2) = \frac{\mathrm{Res}_{q^2=m^2_{B^*}} f^+(q^2)}{m^2_{B^*} - q^2} + \frac{1}{\pi} \int_{t_0}^\infty dt \frac{\mathrm{Im}\, f^+(t)}{t - q^2 - i\epsilon} . \tag{8}$$

The imaginary parts are consisting of all single and multiparticle states with the same quantum numbers as $f^+(t)$ ($f^0(t)$), *i.e.* $J^P = 1^-$ (0^+), thus a multitude of poles and cuts situated above the $B\pi$ production threshold, $t_0 = (m_B + m_\pi)^2$. However, below t_0 (but above the semileptonic $B \to \pi$ region) there is a B^* vector meson ($m_{B^*} = 5.325$ GeV) contributing to the $f^+(t)$ form factor. The residue at that pole is the product of the $B^*B\pi$ coupling and the coupling of B^* to the vector current:

$$\mathrm{Res}_{q^2=m^2_{B^*}} f^+(q^2) = \frac{1}{2} m_{B^*} f_{B^*} g_{B^*B\pi} , \tag{9}$$

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[3] The overall logarithmic correction proportional to $(\alpha_s(m_b)/\alpha_s(\mu))^{-2/\beta_0}$ (arising from the perturbative matching between the QCD and the heavy quark vector current) are irrelevant for our discussion.

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where the standard definitions were used:

$$\langle 0|V_\mu|B^*(p,\epsilon)\rangle = f_{B^*} m_{B^*} \epsilon_\mu \ ,$$

$$\langle B^-(p)\pi^+(q)|B^{*0}(p+q,\epsilon)\rangle = g_{B^*B\pi}(q\cdot\epsilon) \ . \tag{10}$$

While the lattice is providing us with better and better estimate for f_{B^*} [8], the value of $g_{B^*B\pi}$ remains vague. From the above definitions, one immediately finds that $\text{Res}_{q^2=m^2_{B^*}} f^+(q^2)$ scales as $m_B^{3/2}$.

The simple pole dominance would mean that only the first term in the r.h.s. of (8) survives, whereas all the other states which can couple to the vector current and are above t_0, would eventually cancel. Moreover, one would encounter the discrepancy when trying to reconcile the pole dominance ansatz with (6), because the pole dominance gives $f^+(0) \sim m_B^{-1/2}$.

The easiest way out was to suppose the pole behavior for $f^0(q^2)$, and the double pole (dipole) one for $f^+(q^2)$ [9, 10]:

$$f^0(q^2) = \frac{f(0)}{1 - q^2/m^2_{0^+}} \ , \qquad f^+(q^2) = \frac{f(0)}{(1 - q^2/m^2_{1^-})^2} \ . \tag{11}$$

This choice basically satisfies all constraints but it is not informative at all: we know that we have a pole at B^* (and more importantly, we know its position) which we trade for a better fit with data, without getting any supplementary physical information. For that reason, we wanted to propose a new parametrization which would satisfy all the constraints mentioned above, and use the fact that we know where the first pole is. In this way the fit with data will turn to be far more informative.

3. Proposal One should always start from the fact that $f^+(q^2)$ **does have** a pole at $m^2_{B^*}$. The contribution of other higher states (for which we do not know positions, nor residues), can be parametrized as an effective pole whose position is to be determined relatively to the first pole. In other words, we propose:

$$f^+(x) = c_B \left(\frac{1}{1-x} - \frac{\alpha}{1 - \frac{x}{\gamma}} \right) \ , \tag{12}$$

where we introduced $x = q^2/m^2_{B^*}$, and parameters α and γ - positive constants which scale with m_B. Obviously, $c_B m^2_{B^*}$ is the residue of the form factor at B^*. Note that $\gamma m^2_{B^*}$ is the squared mass of an effective 1^- excited $B^{*\prime}$-state, which is equal to $(m_{B^*} + \Delta)^2 \simeq m^2_{B^*} + 2\Delta m_{B^*}$, where Δ does not depend on the heavy quark in the heavy quark limit. Therefore, $\gamma \simeq 1 + 2\Delta/m_{B^*}$. We now check the scaling laws. At q^2_{max}, one has:

$$f^+(x_{max}) \simeq c_B m_B \left(1 + \frac{\alpha\gamma}{(1-\gamma)m_B} \right) \tag{13}$$

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Since c_B scales as $\sim m_B^{-1/2}$, then in order to fulfill the heavy quark scaling law (4) $f^+(x_{max}) \sim m_B^{1/2}$, $1 - \gamma$ can at most scale as $1/m_B$. That is consistent with the above observation on γ. It is also clear that in this counting, $\alpha\gamma$ scales as a constant. On the other hand, at $q^2 = 0$ we have:

$$f^+(0) = c_B(1 - \alpha) \sim m_B^{-1/2}(1 - \alpha) \sim m_B^{-3/2} \quad , \tag{14}$$

which means that $(1 - \alpha)$ scales as $\sim 1/m_B$. This shows why we have chosen the sign "$-$" between the two terms in (12). In fact, it is easy to see that both α and γ scale as $1 + \text{const.}/m_B + \cdots$. To build f^0, we take into account $f^0(0) = f^+(0)$. For convenience, we can measure the position of the effective pole to f^0 in terms of $x = q^2/m_{B^*}^2$, and write

$$f^0(x) = \frac{c_B(1 - \alpha)}{1 - \frac{x}{\beta}} \quad . \tag{15}$$

One can easily see that $f^0(q_{max}^2) \sim m_B^{-1/2}$ as it should, while $f^0(0) \sim m_B^{-3/2}$ by construction. Using the same argument as for γ, it is obvious that β also scales like $1 + \mathcal{O}(1/m_B)$.

These four parameters $(c_B, \alpha, \beta, \gamma)$ contain a considerable information on the process: c_B gives a residue of the form factor at the pole B^*, α measures the contribution of the higher states which are parameterized by an effective pole at $m_{B^*_{\text{eff}}}^2 = \gamma \cdot m_{B^*}^2$. Contributions to the scalar form factor are parameterized by another effective pole whose position is at $m_{B^*_{0 \text{ eff}}}^2 = \beta \cdot m_{B^*}^2$. The situation with the form factor f^+ resembles the situation with the electromagnetic form factor of a nucleon, where the simplest ρ-dominance is not sufficient to describe a dipole behavior of the form factor at large q^2, and one needs to introduce at least one extra excited vector meson.

Present lattice analyses have limited statistical accuracy and accessible kinematical region for the form factors and for a description of lattice data a more constrained parametrization is required. In the limit of large energy of the light meson (large recoils), in the rest frame of the heavy meson

$$E = \frac{m_B}{2}\left(1 - \frac{q^2}{m_B^2} + \frac{m_\pi^2}{m_B^2}\right) \stackrel{m_B \to \infty}{\simeq} \frac{m_B}{2}(1 - x) \quad . \tag{16}$$

In Ref. [7], it was shown that in the heavy quark limit and for large recoils ($m_B \to \infty$ and $E \to \infty$), there exists one nontrivial relation between the two form factors, namely:

$$f^0 = \frac{2E}{m_B} f^+ \quad . \tag{17}$$

In this limit, and by using the above counting argument with $\alpha = 1 - \alpha_0/m_B$, $\gamma = 1 + \gamma_0/m_B$, and $\beta = 1 + \beta_0/m_B$, we have

$$f^+ = \frac{c_0 \, \alpha_0}{2E\sqrt{m_B}} \left[1 - \frac{\gamma_0}{\alpha_0}\left(1 - \frac{m_B}{2E}\right) + \mathcal{O}(1/m_B) + \ldots\right]$$

$$f^0 = \frac{c_0 \, \alpha_0}{2E\sqrt{m_B}} \; [1 + \mathcal{O}(1/m_B) + \ldots] \; . \tag{18}$$

Therefore, to satisfy the relation (17) it is necessary to have $\gamma_0 = \alpha_0$. It is reasonable then to take, $\alpha + \gamma = 2$, but also $\alpha = 1/\gamma$, which satisfy (17) up to terms $\mathcal{O}(1/m_B^2)$. In what follows, we use $\alpha = 1/\gamma$.

To summarize, the recipe is the following one: in lattice analyses, one first calculates m_{B^*} (from the corresponding two-point correlation function)[4] to measure x, and then fits the data according to:

$$\frac{1}{f^0(x)} = \frac{1}{c_B(1-\alpha)} \left(1 - \frac{x}{\beta}\right) \, ,$$
$$\frac{1}{f^+(x)} = \frac{1}{c_B(1-\alpha)} \, (1-x) \, (1-\alpha x) \, , \tag{19}$$

which makes altogether three parameters (c_B, α, β) to perform this fit.

3. Illustration Now, we would like to show how it works on the specific example with the set of lattice data. Before the new (systematically improved) results are released[5], we decided to use the most recent available results, those of JLQCD [11]. Since our purpose

q^2 [GeV^2]	20.7	19.2	18.3	17.4	16.4	15.4
x	0.70	0.65	0.62	0.59	0.55	0.52
$f^0_{B\to\pi}(q^2)$	0.72(3)	0.71(3)	0.67(2)	0.66(4)	0.65(4)	0.60(7)
$f^+_{B\to\pi}(q^2)$	2.09(8)	1.52(8)	1.41(7)	1.28(10)	1.20(12)	1.11(17)

Table 1: *Lattice results obtained by the Hiroshima group, JLQCD* [11].

is only to illustrate the parametrization (19), we will not further comment on the technical details used in the calculation of JLQCD. However, one should note that the lattice groups reported results by assuming the negligible SU(3) breaking effects, which is also supported by the QCD sum rule calculation [6] [12], as well as by experiment with D-mesons [14]. That

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[4]In experiment, one fixes it to the physical mass of the vector meson B^*.

[5]APE and UKQCD groups, are already analyzing their new respective data.

[6]Note that the LCQCD sum rules predict larger SU(3) breaking [13], which is the consequence of the normalization of the light meson wave functions (implying that $f^+_{B\to K}(0)/f^+_{B\to\pi}(0) \simeq f_K/f_\pi$). This is in conflict with the findings of lattice calculations [2]. Further research clarifying that issue is needed.

means that the pion used in the JLQCD calculation is consisted of degenerate quarks of mass near the strange one. Also, the spectator quark in B meson is the "s" quark[7].

First, we fix the mass of $m_{B^*} = 5.46$ GeV, as obtained in the same calculation of JLQCD. In realistic situation, one of course takes the experimental $m_{B^*} = 5.32$ GeV. Then we fit the data with the expressions (19), *i.e.* by minimizing:

$$\chi^2 = \sum_i \left\{ \left(\frac{(1/f^0)_i - \frac{1}{c_B(1-\alpha)}\left(1 - \frac{x_i}{\beta}\right)}{\delta\left(1/f^0\right)_i} \right)^2 + \left(\frac{(1/[f^+(1-x)])_i - \frac{1}{c_B(1-\alpha)}(1 - \alpha x_i)}{\delta\left(1/[f^+(1-x)]\right)_i} \right)^2 \right\}$$

from which we obtain the following result:

$$f^{+,0}(0) = 0.38(8) \ , \quad \alpha = 0.54(17) \ , \quad \beta = 1.4(9) \ . \tag{20}$$

The best fit and the lattice data are depicted in Fig.1. These conservative errors in (20) are estimated from the correlated fit, corresponding to $\Delta\chi^2 = 3.53$ (68% C.L.) [17], which defines the ellipsoid of errors (three parameters). The errors we quote in (20) correspond to the extrema of that ellipsoid. As we already mentioned, by our parametrization, besides the first pole in $f^+(x)$ all the other singularities are replaced by an effective pole, which is situated at $m_{B^*_{\text{eff}}} = m_{B^*}\sqrt{\gamma} = 1.3^{+0.3}_{-0.2} \, m_{B^*}$. The effective pole for $f^0(x)$ is at $m_{B^*_{0\,\text{eff}}} = m_{B^*}\sqrt{\beta} = 1.2^{+0.3}_{-0.5} \, m_{B^*}$.

An important advantage of this parametrization is its transparent physical meaning. For example, after identifying $f(0) = c_B(1-\alpha)$, we may also extract the information on the residue of the form factor f^+ at B^*, *i.e.* about the coupling $g_{B^*B\pi}$:

$$c_B = 0.8(3) \quad \Rightarrow \quad g_{B^*B\pi} = 1.6(6)m_{B^*}/f_{B^*} \quad . \tag{21}$$

If we take the physical value $m_{B^*} = 5.32$ GeV, and $f_{B^*} \simeq 0.2$ GeV, we see that the resulting $g_{B^*B\pi} = 42 \pm 16$, which is compatible with most of today's estimates of this coupling (see Tab.2 in Ref. [3]). A statistical (and more importantly, a systematic) improvement of the lattice data will provide more reliable estimate of this coupling.

The ellipsoid of errors is of the cigar shape so that one can reduce the errors and consequently get some additional information, by making reasonable assumptions. One can for instance make the following exercise. We saw that the position of the *effective* pole of the $f^+(q^2)$ function is at $1.3^{+0.3}_{-0.2} \, m_{B^*}$. If we now fix it to $(1.15, 1.20, 1.25, 1.30, 1.35) \times m_{B^*}$, we obtain the results listed in Tab. 2. We notice that the data tend towards larger values of the mass of the scalar effective pole as compared to the expected mass of the lowest scalar meson [8], $\beta \simeq 1.17$, though they are consistent with this value. Note that an expected mass of the first excited vector state according to theoretical estimates should be close to 6 GeV, which again corresponds to a smaller value of $\gamma \approx 1.3$ (or $\alpha \approx 0.7$), but within the errors agrees with the values found in our analysis.

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[7]To our knowledge, among recent studies, only the APE collaboration [15] made the extrapolation to the chiral limit.

[8]The scalar (0^+) meson is expected in the region of the measured bump [16] of B^{**} states: $m_{B^{**}} = 5.73$ GeV.

$m_{B^*_{\text{eff}}}/m_{B^*}$	1.15	1.20	1.25	1.30	1.35
α	0.76	0.69	0.64	0.59	0.55
$f(0)$	0.30(1)	0.32(1)	0.34(1)	0.36(1)	0.38(1)
$g_{B^*B\pi}f_{B^*}/m_{B^*}$	2.45(5)	2.10(4)	1.90(4)	1.76(4)	1.65(4)
β	1.12(6)	0.19(7)	1.26(8)	1.32(9)	1.39(10)
$m_{B^*_0\ \text{eff}}/m_{B^*}$	1.06(3)	1.09(3)	1.12(4)	1.15(4)	1.18(4)
f_B/f_π	2.5 ± 1.0	1.8 ± 0.6	1.6 ± 0.4	1.4 ± 0.3	1.3 ± 0.3

Table 2: *Results of fits for a fixed γ. The last row is obtained using Eqs. (5) and (15) in which we took $x = 0.983$.*

Now, when we have fixed the parameters by the lattice data (20), we would like to compare the resulting shape of the form factor $f^+(q^2)$ to the result of the light cone QCD sum rule, which is applicable in the low q^2 region. To do that we define $f^+(q^2)/f^+(0)$, and plot them both in Fig. 2. We see that our parametrization, constrained by the lattice data at large q^2's, indeed reproduces the shape of the form factor at low q^2's as predicted by QCD sum rule. To be consistent, in the comparison we used the physical $m_{B^*} = 5.32$ GeV.

4. Summary   In this letter we proposed the physically more informative way to fit the data for the form factors in heavy to light decays. This is illustrated on the example of $B \to \pi$ semileptonic decay, and can be also used in $D \to \pi$, as well as in the other *heavy $\to$ light* transitions. Initially, the parametrization for two form factors contains four parameters (12),(15), which we further reduced to three (19). All parameters scale with the heavy mass as const. $+ \mathcal{O}(1/m_B)$, except for c_B. The proposed parametrization gives us the information on the singularities which decisively influence the shape of the form factor in the semileptonic region, and allows us to extract the value of the residue at the nearest pole (*i.e.* to determine the value of $g_{B^*B\pi}$-coupling).

On the specific example, we have shown that when fixing the parameters in our parametrization by the lattice data in the large q^2 region, one reproduces (to a good precision) the shape of the form factor obtained by QCD sum rules at low and intermediate q^2's.

We believe that this parametrization may be useful in the forthcoming analyses of systematically improved lattice data, as well as in experimental studies of exclusive *heavy $\to$ light* decays.

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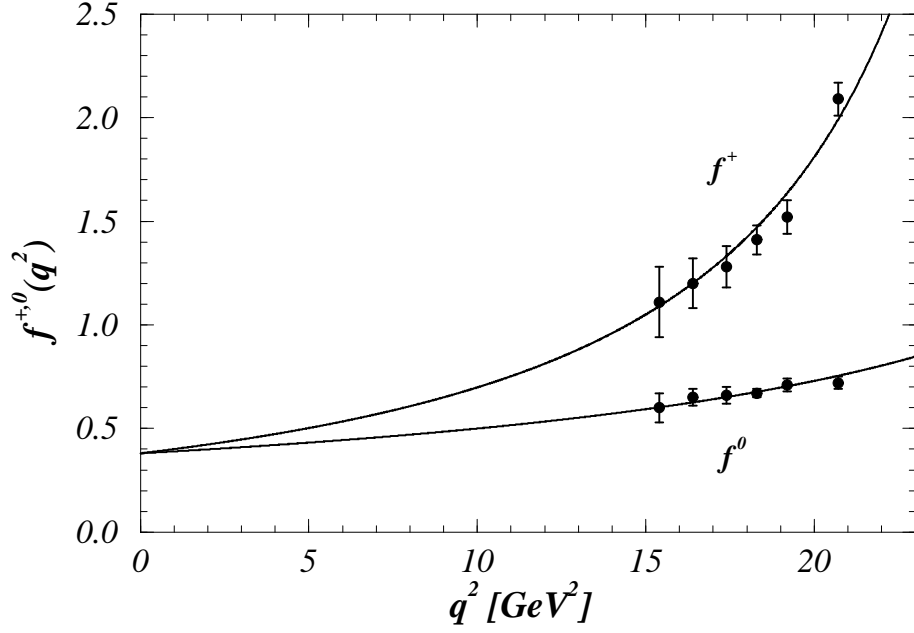[18] E. Bagan, P. Ball, V.M. Braun, Phys. Lett. **B417** (1998) 154.

Figure 1: *Fitting the lattice data using the parametrization* (19). *Note that the fit of f^+ form factor is constrained by the precise data for f^0. For easiness, only the central curves (without errors in parameters) are displayed.*
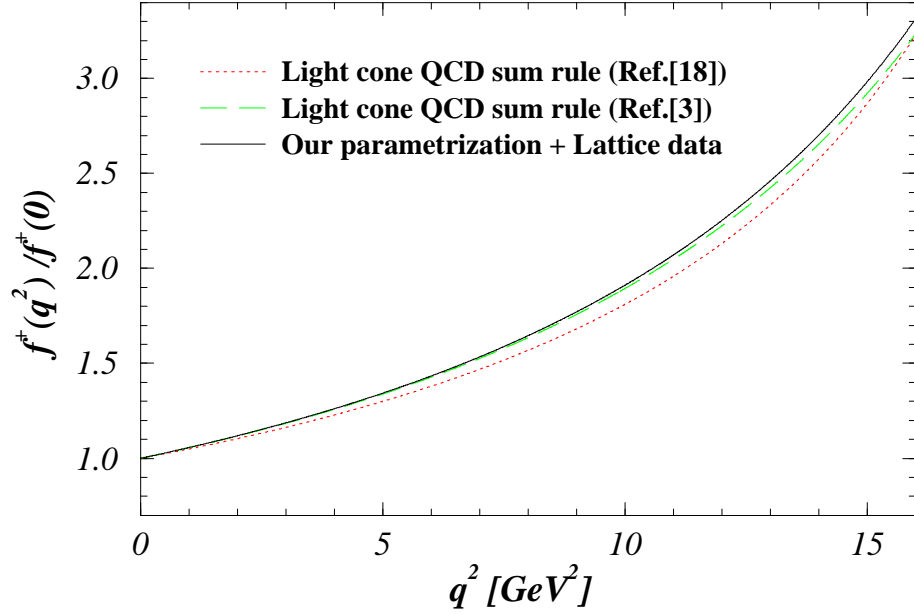

Figure 2: *The shape of the form factor from our parametrization* (19), *constrained by the lattice data at large q^2, is compared to the QCD sum rule predictions at "low" q^2 (Eq.(132) in Ref. [3] for the dashed curve, and Eq.(23) in Ref. [18] for the dotted curve). A very good agreement is found.*